zero shift, which indicates that the signal matches perfectly at zero lag.

Properties of Correlation JAN 18.

$$r_{xy}[l] = \sum_{n=-\infty}^{\infty} x[n]\,y[n-l] = x[l] * y[-l]$$

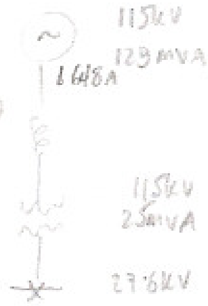
$$r_{yx}[l] = \sum_{n=-\infty}^{\infty} y[n]\,x[n-l] = y[l] * x[-l]$$

$V = 1.0\,pu$

$I_{BASE} = \frac{MVA_{base}}{(\sqrt{3})(V_{base})}$

$I_{F_{pu}} = \frac{648A}{I_{BASE}}$ $j0.36$

$j0.055$



$\therefore r_{xy}(l) = r_{yx}[-l]$; $r_{xy}(l)$ is simply a folder seq. of $r_{xy}(l)$.

Thus, they provide the same information regarding the distance of the target.

Ex Determine the cross-correlation sequence $r_{xy}[l]$ of the following sequences

2 here...

$$x[n] = \{2, -1, 3, 7, \underset{\uparrow}{1}, 2, -3\}$$
$$\scriptstyle -4 \quad -3 \quad -2 \quad -1 \quad n=0 \quad 1 \quad 2$$

$$y[n] = \{1, -1, 2, -2, \underset{\uparrow}{4}, 1, -2, 5\}$$
$$\scriptstyle -4 \quad -3 \quad -2 \quad -1 \quad n=0 \quad 1 \quad 2 \quad 3$$

... 4 here ...

overlaps for 6 shifts based on 2 & 4.
→ no overlap for $l \geq 7$

Sol'n

$$r_{xy}[l] = \sum_{n=-\infty}^{\infty} x[n]\,y[n-l]$$

$l \geq 7 : r_{xy}[l] = 0$

+ve side: $l \geq 0$

$l = 0$: $r_{xy}[0] = \sum_{n=-\infty}^{\infty} x[n]\,y[\overset{n-l}{n}] = \sum\{2, 1, 6, -14, 4, 2, 6, 0\} = 7$

$l = 1$: $r_{xy}[1] = \sum_{n=-\infty}^{\infty} x[n]\,y[n-\overset{l=1}{1}] = \sum\{0, -1, -3, 14, -2, 8, -3\} = 13$

(2×0) (−1)(1) (3×−1) (7×2) (1×−2)

complete for $l = 2$ to $l = 6$.

−ve side $l < 0$:

shifts left.

$l = -1$: $r_{xy}(-1) = \sum_{n=-\infty}^{\infty} x[n]\,y[n+1] = \sum\{-2, -2, -6, 28, 1, -4, -15\} = 0$.

for $l \leq 8$, $r_{xy}[l] = 0$ $\quad\therefore$ only calculate for $l = -2$ to $l = -7$.

FINAL RESULT: $r_{xy}[l] = \{10, -9, 19, 36, -14, 33, 0, 7, 13, -18, 16, -7, 5, -3\}$

Significance: If $x[n]$ is transmitted signal and $y[n]$ the reflected signal there could be objects detected and the distance of the objects could be determined from the delays corresponding highest or closest sample values 33 & 36.

1.6. Stability and causality tests of an LTI system

can draw graphs.

→ Stability : $y[n] = \sum_k x[k]h[n-k] = \sum_k x[n-k]h[k]$

If $|x[n]| < \infty$ then for a stable system $|y[n]| < \infty$

$\therefore$ For stable systems $\sum_{k=-\infty}^{\infty} |h[k]| < \infty \Rightarrow \sum_{n=-\infty}^{\infty} |h[n]| < \infty$

→ Causality : for a causal system $y[n_0]$ depends on $x[n]$ for $n \leq n_0$.
↳ non-anticipative.

$y[n_0] = \sum_{n=-\infty}^{\infty} x[n_0-k]h[k]$ $\therefore$ for causal system :

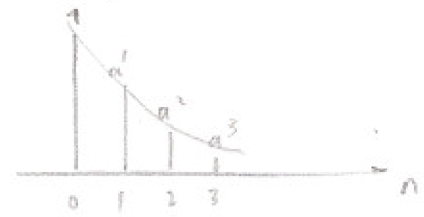
$\Rightarrow y[n_0] = \sum_{k=0}^{\infty} x[n_0-k]h[k]$ (NON CAUSAL SYSTEMS CANNOT BE IMPLEMENTED IN REAL TIME)

$\therefore$ for causal systems, $h[k] = 0$, $k<0$ $\Rightarrow$ $h[n]=0$, $n<0$

Ex The impulse response of a system $h[n] = a^n u[n]$. Test the causality and stability of this system. $(|a|<1)$

Sol'n:

$h[n] = a^n u[n]$



$a^0 + a^1 + a^2$

For stability $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$ $\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} |a|^n = \frac{1}{1-|a|} < \infty$ $\therefore$ stable.

Since $h[0]=1$ is the max. value, and $a^n \to 0$ as $n \to \infty$, this condition is satisfied.

Causality : must be causal due to $u[n]$ function
step → non-zero for $n \geq 0$.

1.7. Linear constant coefficient difference eq'n (DE)
(Similar to differential eq'n)

An n^{th} order linear const. coeff D.E. can be written as

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_k x[n-m]$$

$$\Rightarrow y[n] = \sum_{m=0}^{M} \frac{b_k}{a_0} x[n-m] - \sum_{k=1}^{N} \frac{a_k}{a_0} y[n-k]$$

Thus, a 2^{nd} order D.E. can be written as

$$y[n] = y[n-2] + 2y[n-1] + x[n]$$

→ The order is determined by the highest delay in the output sequence.

The solution of a D.E. is similar to the solution of a differential eqn.

1.7.1: Recursive computation method.
↑ feedback present

Ex Solve the following first order diff. eq'n using the recursive method when

$$x[n] = k\delta[n] \quad \text{and} \quad y[-1] = c \quad \text{(boundary cond'n)}.$$

Sol^n:

$$y[n] = a\,y[n-1] + x[n] \cdots ①$$

$n \geq 0$: $y[0] = a\,y[-1] + x[0] = ac+k$

$$y[1] = a\,y[0] + \underbrace{x[1]}_{=0\ (\delta[n]\ \text{function})} = a(ac+k) + 0 = a^2c + ak$$

$$y[2] = a\,y[1] + \underbrace{x[2]}_{=0} = a^3c + a^2c$$

$\vdots$

$$y[n] = a^{n+1}c + a^n k, \quad n \geq 0$$

$n<0$: Must modify eq'n ① to accomodate the given boundary condition.

From ① : $y[n-1] = \frac{1}{a}\left(y[n] - x[n]\right)$

$n=-1$ $\quad y[\overbrace{-1-1}^{-2}] = \frac{1}{a}\left(y[-1] - \cancel{x[-1]}\right) = a^{-1}c$

$n=-1$ $\qquad\qquad 0 \Rightarrow \delta[n]$ function

$n=-2$ $\quad y[-3] = \frac{1}{a}\left(y[-2] - \cancel{x[-2]}\right) = a^{-2}c$

$\qquad\qquad\qquad 0$

$$y[n] = a^{n+1}\cdot c \quad , \quad n<0$$

$\therefore$ the general solution, $y[n] = a^{n+1}c + ka^{n}u[n]$

$= y_h[n] + y_p[n]$

Since this is only applicable to $n>0$.

homogeneous (transient) Solution

particular (steady-state) Solution.

-- In general, a D.E. is a type of LTI system. (depends on initial conditions).

NOTES:

① The system was implemented by computing the output $y[n]$ in both directions (+ve & -ve) of n. This procedure is non-causal.

② If $k=0$, $x[n]=0$, but $y[n] = a^{n+1}c$ (there is an output even though there is no input) $\therefore$ the system is non-linear

③ If $x[n]$ is shifted by n_0 samples, then $x[n] = k\delta[n-n_0]$ then $y[n] = ca^{n+1} + ka^{n-n_0}u[n-n_0]$. Thus, the system is not time-invariant

④ $y[n]=h[n] = ca^{n+1} + a^{n}u[n]$ when $x[n] = \delta[n]$ (with $k=1$)

⑤ Since $h[n] \neq 0$ for $n<0$, the system is non-causal.

A linear constant coefficient D.E. is further specified to be linear, time invariant and causal if the auxiliary condition is that

$$\left.\begin{array}{ll} \text{if } x[n]=0, & n\leq n_0 \\ \text{then } y[n]=0, & n\leq n_0 \end{array}\right\} \text{ initial rest condition.}$$

1.7.2: Conventional or Traditional Method.

— $y_p[n]$ can be found from the z-transform
$y_h[n]$ " " " " " homogeneous eqn as follows.

From previous example,

$$\begin{cases} \frac{d^2y}{dt^2} + \frac{2dy}{dt} + 3 = 0 \\ AE: \; m^2 + 2m + 3 = 0 \end{cases}$$

$m_1, m_2 \Rightarrow y(t) = A_1 e^{m_1 t} + A_2 e^{m_2 t}$

homogeneous eq^n $y[n] - a y[n-1] = 0$

Let $y[n] = c^n$ be a trial solution

$\therefore c^n - ac^{n-1} = 0 \Rightarrow (1 - ac^{-1}) c^n = 0$ $\therefore 1 - ac^{-1} = 0$ — auxiliary eqn.

$c^n \neq 0$

$\Rightarrow c = a = c_1$

$\therefore y_h[n] = A_1 c_1^n = A_1 a^n$

repeated roots: $m_1 = m_2 = m$

$y_h(t) = Ae^{mt} + Ate^{mt}$

discrete time is $\underline{n}$.

— In general form: $y_h[n] = A_1 c_1^n + A_2 c_2^n + A_3 c_3^n$

$\therefore$ The general sol'n is $y[n] = y_h[n] + y_p[n]$

$$y[n] = A_1 a^n + k a^n u[n]$$

Initial condition is then applied: $y[-1] = c$

$$\therefore y[-1] = c = A_1 a^{-1} + 0$$

$$\Rightarrow A_1 = ac$$

$$\therefore \boxed{y[n] = c a^{n+1} + k a^n u[n]}$$

as found before in § 1.7.1.

1.8 Frequency Domain Representation of Discrete Time Signals & Systems.

Let $x[n] = e^{j\omega_0 n} = \cos\omega_0 n + j\sin\omega_0 n$

$$x[n] = e^{j\omega_0 n} \longrightarrow \underset{h[n]}{\boxed{LTI}} \longrightarrow y[n] = e^{j\omega_0 n} H(e^{j\omega})$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]\,h[k] = \sum_{k=-\infty}^{\infty} e^{j\omega_0(n-k)} h[k]$$

$$= e^{j\omega_0 n} \sum_{k=-\infty}^{\infty} e^{-j\omega_0 k} h[k] = H(e^{j\omega_0})\, e^{j\omega_0 n},$$

where $$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]\, e^{-j\omega k}$$ ← the freq. domain representation of $h[n]$.

aka. freq. response of system

$e^{j\omega_0 n}$: eigenfunction

$H(e^{j\omega})$: eigen value of the system. corresponding

Therefore, the freq. domain representation of $x[n]$ is given by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

\# $H(e^{j\omega})$ is a periodic function of period 2π.

$$H(e^{j(\omega+2\pi)}) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j(\omega+2\pi)n}$$

$$\boxed{e^{-j2\pi n} = 1}$$

$$= \sum_{n} h[n]\, e^{-j\omega n}$$

$$= H(e^{j\omega})$$ i.e.

Ex Find the frequency response of the following LTI system:

$$y[n] = x[n-n_d]$$

Sol^n:

Method 1: $x[n] = e^{j\omega n}$ $\therefore$ $y[n] = e^{j\omega(n-n_d)} = e^{j\omega n}\cdot e^{-j\omega n_d}$

$$y[n] = e^{j\omega n} H(e^{j\omega})$$

$\therefore \; H(e^{j\omega}) = \boxed{e^{-j\omega n_d}}$

Method 2:

Let $x[n] = \delta[n]$, then $y[n] = h[n] = \delta[n-n_d]$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \underbrace{\delta[n-n_d]}_{=1 \text{ when } n=n_d} e^{-j\omega n}$$

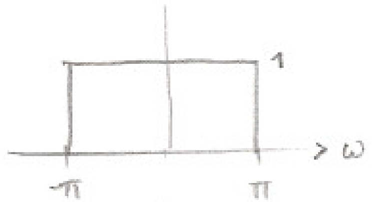
$$\Rightarrow H(e^{j\omega}) = \boxed{e^{-j\omega n_d}}$$

Magnitude response : $|H(e^{j\omega})|$ vs. ω

Phase response : $\angle H(e^{j\omega})$ vs. ω

← discrete bode plot.

$|H(e^{j\omega})| = 1$:



$\angle H(e^{j\omega}) = -\omega n_d$: